Fluctuations around classical solutions for gauge theories in Lagrangian and Hamiltonian approach

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Abstract

We analyze the dynamics of gauge theories and constrained systems in general under small perturbations around a classical solution in both Lagrangian and Hamiltonian formalisms. We prove that a fluctuations theory, described by a quadratic Lagrangian, has the same constraint structure and number of physical degrees of freedom as the original non-perturbed theory, assuming the non-degenerate solution has been chosen. We show that the number of Noether gauge symmetries is the same in both theories, but that the gauge algebra in the fluctuations theory becomes Abelianized. We also show that the fluctuations theory inherits all functionally independent rigid symmetries from the original theory, and that these symmetries are generated by linear or quadratic generators according to whether the original symmetry is preserved by the background, or is broken by it. We illustrate these results with the examples.

1 Introduction

Dynamics of linearized perturbations, obeying the equations of motion of the quadratic action formulated around a classical solution (background) of a field theory, has been widely used for numerous applications.[1] Thus, it is used as a test of stability, where the fluctuations around a stable solution —i.e., vacuum— have harmonic oscillator dynamics. In general, the quadratic potentials also provide quantum corrections for an "effective" mass of a solution, which can be a wave packet such as a soliton, or can identify tachyonic modes (with negative square mass) which would signal an instability, and so on. Let us emphasize that here we perturb only fundamental fields in a theory, without making any expansion in the coupling constant, nor the quantum loop expansion in $\hbar$. These, and many other numerous uses of the fluctuations theory, such

[1] We use indistinctly the words perturbations and fluctuations, the latter one corresponding more closely to the language usual in quantum theory and statistical mechanics.

as quantization of solitons, spontaneous breaking of symmetry, gravitational waves, etc., have become so standard in physics that they can be found in any textbook on field theory.

Intuitively, the dynamics of the original and "linearized" (described by a quadratic action) theories should exhibit some parallelisms. In gauge systems, however, the presence of unphysical degrees of freedom, and the frequent appearance of constraints, obscures this intuition. For example, there are systems which seem to have more degrees of freedom when linearized around some backgrounds [1]. In these quadratic theories, the gauge symmetry appears as broken with respect to the full theory.

In addition, and connected with the previous observation, one may ask what is a criterion for the Legendre transformation to commute with the process of getting the theory of quadratic fluctuations in both Lagrangian and Hamiltonian approaches.

Other questions that may arise are whether the quadratic action contains the same rigid and gauge symmetries as the original one, how its constraint structure looks like and how the corresponding canonical theory is formulated.

In this paper we address all these questions. We write out a criterion that guarantees that the quadratic action contains as much gauge freedom as the original one. In fact we show that the gauge algebra, if it was originally non-Abelian, in the fluctuations theory becomes Abelianized. This agrees with the fact that a non-Abelian theory cannot be described by a quadratic action, but it requires higher-order terms.

We also show that the canonical quadratic Hamiltonian for the fluctuations is built up from two different pieces of information: one is obviously the quadratic term of the expansion of the original canonical Hamiltonian, whereas the other, not so obvious, consists in the quadratic terms of the expansion of the original primary constraints. This result is quite natural from the viewpoint of the Dirac-Bergman theory of constrained systems, on which we rely throughout the paper. In connecting the Lagrangian and the Hamiltonian formulations at the level of the original action with those at the level of the quadratic action, we see that a mismatch appears between the respective Legendre maps (from tangent space to phase space), but that such mismatch is of higher order in the fluctuations and thus does not affect the consistency of our procedure.

Concerning the constraint algorithm in the fluctuations theory —either in the Lagrangian, or in the Hamiltonian formulation— it is shown that it reflects the structure of the algorithm that holds for the original theory. This result is not a priori obvious, because when there is more than one generation of constraints, that is, when new constraints arise from evolution of original primary constraints, then the process of truncation of higher order terms may not commute with taking the time derivative and Poisson bracket. In particular, we show that the original Second Class constraints yield Second Class constraints for the fluctuations theory, and First Class constraints yield *Abelianized* First Class constraints.

Noether symmetries and conserved quantities are also shown to be inherited from the original theory to the fluctuations theory, but in a non-straightforward way. In fact, due to the presence of the classical solution –the background– the original Noether symmetries split according to

whether they are preserved by the background, or are broken by it. Remarkably, it turns out that those that are respected by the background yield rigid symmetries for the quadratic action with *quadratic* generators, whereas the broken symmetries yield symmetries with *linear* generators. The importance of quadratic generators is noteworthy in field theories with supersymetry where a BPS state, a solution which preserves some supersymmetries, is preferred as a ground state since it plays significant role in the stability of a theory.

Regarding quantization around the classical solution, note that the gauge fixing is technically simpler for the quadratic theory than for the original theory, owing to the Abelian structure of the new gauge group. This means that one could have spared the technicalities of gauge fixing in the non-Abelian case and proceed instead to the easier gauge fixing for the quadratic –Abelian– theory around the classical solution.

After introducing some notation in the next section, we address the tangent space version and the canonical version of the fluctuations theory in Sections 3 and 4, where the connection between both formalisms is analyzed as well as their constraint algorithms. In Section 5 we study the Noether symmetries for the fluctuations theory. Examples are discussed in Section 6, and Section 7 is devoted to conclusions.

2 Notation

We will use for simplicity the language of mechanics. Since our prime interest are gauge field theories, a quick switch to the field theory language can be achieved by using DeWitt's condensed notation [2].

Consider the dynamics of a classical mechanical system with finite number of degrees of freedom, described by a Lagrangian $L(q,\,\dot{q})$ depending at most on first derivatives, up to divergence terms, and which does not depend on time explicitly (first-order systems). The local coordinates q^i $(i=1,\ldots,n)$ parameterize a configuration manifold $\mathcal{Q}$ of dimension n, and therefore the entire dynamics of the system happens on the corresponding tangent bundle $T\mathcal{Q}$ which is a configuration-velocity space $(q,\,\dot{q})$. The Euler-Lagrange equations of motion (e.o.m.) are[2]

$$[L]_i := \alpha_i - W_{ij}\,\ddot{q}^j = 0\,,$$

with the Hessian matrix

$$W_{ij} \equiv \frac{\partial^2 L}{\partial\dot{q}^i\partial\dot{q}^j}\,, \tag{1}$$

and

$$\alpha_i := -\frac{\partial^2 L}{\partial\dot{q}^i\partial q^j}\,\dot{q}^j + \frac{\partial L}{\partial q^i}\,.$$

Regular systems have invertible Hessian. We are, however, interested in singular Lagrangians with non-invertible Hessian matrices, also called constrained systems [3]–[7], since the gauge theories rely on them.

[2] All functions are assumed to be continuous and differentiable as many times as the formalism requires.

In order to pass to the Hamiltonian formalism, we apply the Legendre map $\mathcal{F}L: \; T\mathcal{Q} \to T^*\mathcal{Q}$ to the original theory, which maps configurational space into phase space,

$$(q,\dot{q}) \to (q, p=\hat{p}(q,\dot{q}))\,,$$

where the momentum map is

$$\hat{p}(q,\dot{q}) := \frac{\partial L}{\partial \dot{q}}\,.$$

We made the assumption that the rank of the Hessian matrix is constant everywhere. If this condition is not satisfied throughout the whole tangent bundle, we restrict our considerations to a region of it, with the same dimensionality, where this condition holds. For degenerate systems with non-constant rank of W, see [8]. So we are assuming that the rank of the Legendre map $\mathcal{F}L$ from the tangent bundle $T\mathcal{Q}$ to the cotangent bundle $T^*\mathcal{Q}$ is constant throughout $T\mathcal{Q}$ and equal to, say, $2n-k$. The image of $\mathcal{F}L$ is locally defined by the vanishing of k independent functions, $\phi_\mu(q,p), \mu = 1,2,..,k$. These functions are the *primary constraints*, and their pullback $\mathcal{F}L^*\phi_\mu$ to the tangent bundle is identically zero:

$$(\mathcal{F}L^*\phi_\mu)(q,\dot{q}) := \phi_\mu(q,\hat{p}) = 0\,, \quad \forall\, q,\dot{q}\,. \tag{2}$$

The primary constraints form a generating set of the ideal of functions that vanish on the image of the Legendre map. With their help it is easy to obtain a basis of null vectors for the Hessian matrix [9]. Indeed, applying $\frac{\partial}{\partial \dot{q}}$ to (2) we get

$$W_{ij}\left(\frac{\partial \phi_\mu}{\partial p_j}\right)_{|p=\hat{p}} = 0\,, \quad \forall\, q,\dot{q}\,.$$

The basis of null vectors γ_μ, with components γ^j_μ, is denoted as

$$\gamma^j_\mu := \mathcal{F}L^*\frac{\partial \phi_\mu}{\partial p_j}\,. \tag{3}$$

Working with this basis proves to be an efficient way to obtain results for the Lagrangian tangent space formulation by use of Hamiltonian techniques.

3 Expanding the Lagrangian around a classical solution

Denote the solution by q^o. We have assumed that q^o is non-degenerate, i.e., the equations of motion $[L]$ have simple zeroes in $q = q^o$. Although this condition is fulfilled for the most of solutions in various models, there are Lagrangians with degenerate solutions leading, for example, to ineffective (irregular) constraints [10, 11, 12].

In the Hamiltonian formalism, the criterion to have all constraints effective or functionally independent in the vicinity of the solution q^o, is that their Jacobian in the phase space (q,p) evaluated at (q^o,p^o), has maximal rank [13]. This condition can be generalized to Lagrangian

formalism. The n Euler-Lagrange equations $[L]$, containing the evolution equation (with non-vanishing Hessian) generally imply the existence of primary Lagangian constraints (we introduce them below). Typically, its preservation will yield new, secondary constraints, then tertiary, and so on. In order to have full control of the quadratic fluctuations theory around a solution q^o we will require that (i) the rank of the Hessian matrix be constant, (ii) the equations of motion $[L]$ to have simple zeroes in $q = q^o$ (the solutions are non-degenerate) and (iii) that all constraints are effective in a neighborhood of q^o (that is, its Jacobian with respect to the tangent space coordinates be of maximum rank). Note that some of these requirements may cease to hold only in one singular point, which can therefore pass unnoticed if the given conditions are not explicitly checked at this point. This happens in Chern-Simons gauge theories, which have been discussed in Hamiltonian formalism in [8, 11, 12].

Now we introduce the fluctuations theory Lagrangian. First define the fluctuations Q by means of

$$q = q^o + \epsilon Q\,, \qquad (\Rightarrow \dot{q} = \dot{q}^o + \epsilon \dot{Q})\,, \tag{4}$$

with ϵ a small constant parameter, and expand

$$\begin{aligned} L(q, \dot{q}) &= L(q^o, \dot{q}^o) + \epsilon (Q \frac{\partial L}{\partial q}|_o + \dot{Q} \frac{\partial L}{\partial \dot{q}}|_o) + \epsilon^2 \tilde{L}(Q, \dot{Q}; t) + \mathcal{O}(\epsilon^3) \\ &= L(q^o, \dot{q}^o) + \epsilon \frac{d}{dt} (Q \frac{\partial L}{\partial \dot{q}}|_o) + \epsilon^2 \tilde{L}(Q, \dot{Q}; t) + \mathcal{O}(\epsilon^3)\,, \end{aligned} \tag{5}$$

where[3] we have generically denoted $A(q, \dot{q})|_o = A(q^o, \dot{q}^o)$, recalled that $[L]|_o = 0$ (the omitted indices are saturated in an obvious way), and defined the quadratic Lagrangian for small fluctuations[4]

$$\tilde{L}(Q, \dot{Q}; t) := \frac{1}{2}\Big(Q \frac{\partial^2 L}{\partial q \partial q}|_o Q + 2Q \frac{\partial^2 L}{\partial q \partial \dot{q}}|_o \dot{Q} + \dot{Q} \frac{\partial^2 L}{\partial \dot{q} \partial \dot{q}}|_o \dot{Q}\Big)\,. \tag{6}$$

(Since this Lagrangian leads to linear equations of motion, it is just the Lagrangian for linearized fluctuations.) In general $\tilde{L}$ will be time dependent because the solution $q^o(t)$ depends on time explicitly. However, in order not to burden the notation, from now on this time dependence will not be made explicit in the arguments of our functions. Note that the Hessian matrix for $\tilde{L}$ coincides with the Hessian matrix for L computed on the solution q^o, $W|_o$. If we now perform a change of variables $q \to Q$ (see the discussion on the change of variables in Appendix A), noticing that $\frac{\partial}{\partial q} = \frac{1}{\epsilon} \frac{\partial}{\partial Q}$ and $\frac{\partial}{\partial \dot{q}} = \frac{1}{\epsilon} \frac{\partial}{\partial \dot{Q}}$ and applying (5), for the Euler-Lagrange equations we obtain

$$\begin{aligned} [L(q, \dot{q})]_q &= \frac{1}{\epsilon} [L(q^o + \epsilon Q\,, \dot{q}^o + \epsilon \dot{Q})]_Q = \frac{1}{\epsilon} (\epsilon^2 [\tilde{L}(Q, \dot{Q})]_Q + \mathcal{O}(\epsilon^3)) \\ &= \epsilon\, [\tilde{L}(Q, \dot{Q})]_Q + \mathcal{O}(\epsilon^2)\,. \end{aligned} \tag{7}$$

[3]Note that in field theory the second term in (5) is a divergence.

[4]The size of the fluctuations depends not only on the values of the Q variables, but also on the "small" parameter ϵ, which has been factored out from the fluctuations theory.

Thus we see that *if $Q(t)$ is a solution of the Euler-Lagrange e.o.m. for $\tilde{L}$, then $q(t) := q^o(t) + \epsilon Q(t)$ is a solution of the e.o.m. for the original Lagrangian L up to terms of order ϵ^2.*

It can be verified that all quadratic first order Lagrangians (which in general contain linear terms) are equivalent to their own fluctuations theories. This can be seen by making the coordinate transformation $q = q^o + q'$, where q^o is a particular solution of the e.o.m., which transforms the quadratic Lagrangian $L(q, \dot{q})$ into a homogenous function $L(q^o, \dot{q}^o) + L_{hom}(q', \dot{q}')$ (up to a total derivative), where L_{hom} stands for the (second order) homogenous part of L. Using the property of homogenous functions of second degree, $L_{hom}(\epsilon Q) = \epsilon^2 L_{hom}(Q)$, we conclude that the fluctuations Lagrangian is the homogenous part of the original Lagrangian, $\tilde{L}(Q, \dot{Q}) = L_{hom}(Q, \dot{Q})$. This is valid exactly (for any ϵ). The fluctuations Lagrangian and the original quadratic Lagrangian have, therefore, equivalent dynamical structures, including both gauge and rigid symmetries.

3.1 Lagrangian constraints

The equations $[L]$ can be separated into the evolution equations, and the constraints. Taking into account that $W\gamma_\mu = 0$ identically, the primary Lagrangian constraints for L are

$$\chi_\mu := [L]\gamma_\mu = (\alpha - W\ddot{q})\gamma_\mu = \alpha\gamma_\mu \simeq 0\,, \tag{8}$$

where $\simeq 0$ means "vanishing on shell", that is, when the e.o.m. are satisfied. If we expand them under $q = q^o + \epsilon Q$ we get (note that $\chi_\mu(q^o, \dot{q}^o) = 0$ because of $[L]|_o = 0$),

$$\begin{aligned}
\chi_\mu &= (\alpha\gamma_\mu)(q, \dot{q}) = \epsilon(Q\,\frac{\partial(\alpha\gamma_\mu)}{\partial q}|_o + \dot{Q}\,\frac{\partial(\alpha\gamma_\mu)}{\partial \dot{q}}|_o) + \mathcal{O}(\epsilon^2) \\
&= \epsilon(Q\,\frac{\partial([L]\gamma_\mu)}{\partial q}|_o + \dot{Q}\,\frac{\partial([L]\gamma_\mu)}{\partial \dot{q}}|_o) + \mathcal{O}(\epsilon^2) \\
&= \epsilon(Q\,(\frac{\partial[L]}{\partial q}\,\gamma_\mu)|_o + \dot{Q}\,(\frac{\partial[L]}{\partial \dot{q}}\,\gamma_\mu)|_o) + \mathcal{O}(\epsilon^2) =: \epsilon\tilde{\chi}_\mu + \mathcal{O}(\epsilon^2)\,.
\end{aligned} \tag{9}$$

This result suggests that

$$\tilde{\chi}_\mu := Q\,(\frac{\partial[L]}{\partial q}\,\gamma_\mu)|_o + \dot{Q}\,(\frac{\partial[L]}{\partial \dot{q}}\,\gamma_\mu)|_o \simeq 0 \tag{10}$$

are the primary Lagrangian constraints for the theory derived from $\tilde{L}$. Now we will prove this claim.

To this end, notice that we can expand directly $[L(q, \dot{q})]$,

$$[L(q, \dot{q})]_q = [L(q, \dot{q})]_{q^o} + \epsilon\,(Q\,\frac{\partial[L]}{\partial q}|_o + \dot{Q}\,\frac{\partial[L]}{\partial \dot{q}}|_o + \ddot{Q}\,\frac{\partial[L]}{\partial \ddot{q}}|_o) + \mathcal{O}(\epsilon^2)\,,$$

and use that $[L(q, \dot{q})]_{q^o} = 0$ by definition. Then, comparing the above expression with (7), we conclude that the Euler-Lagrange e.o.m. for $\tilde{L}$ can be written as

$$[\tilde{L}(Q, \dot{Q})] = Q\,\frac{\partial[L]}{\partial q}|_o + \dot{Q}\,\frac{\partial[L]}{\partial \dot{q}}|_o - W|_o\ddot{Q}\,.$$

Thus, using the fact that $\gamma_\mu|_o$ are the null vectors of the Hessian $W|_o$ for $\tilde{L}$, the primary Lagrangian constraints for $\tilde{L}$ are

$$\Big([\tilde{L}(Q,\dot{Q})]\gamma_\mu\Big)|_o = (Q\,\frac{\partial[L]}{\partial q}|_o + \dot{Q}\,\frac{\partial[L]}{\partial \dot{q}}|_o)\gamma_\mu|_o = \tilde{\chi}_\mu\,,$$

which coincides with the result in (10) and proves the claim.

The number of the constraints $\tilde{\chi}_\mu$ ($\mu = 1,\ldots,k$) is the same as the number of the original constraints χ_μ since we are dealing with effective constraints, for which the Jacobian $\frac{\partial(\chi_1,\ldots,\chi_k)}{\partial(q,\dot{q})}|_o$ has be non degenerate (has rank k). From this, it follows immediately that $\tilde{\chi}_1,\ldots,\tilde{\chi}_k$ are linearly independent.

4 The canonical formalism

This result $\chi_\mu = \epsilon\tilde{\chi}_\mu + \mathcal{O}(\epsilon^2)$ makes one suspect that the full algorithm of constraints for the original theory will be reproduced, step by step, within the theory of linearized fluctuations. On the other hand we know that the Lagrangian and Hamiltonian constraint algorithms are deeply related, see [9, 14], in the sense that, step by step, one can determine a subset of the Lagrangian constraints as pullbacks —under the Legendre map— of the Hamiltonian constraints, and the rest from the canonical determination of some of the arbitrary functions that appear as Lagrange multipliers in the Dirac Hamiltonian. Since the analysis of the constraint algorithm in the canonical formalism is facilitated by the presence of the Poisson bracket structure, we now turn to the canonical analysis.

If we use the change of variables $q \to Q$, then $\hat{p}$ becomes $\hat{p} = \frac{1}{\epsilon}\frac{\partial L}{\partial \dot{Q}}$ and, using the expansion (5), we obtain

$$\hat{p} = \frac{1}{\epsilon}\Big(\epsilon\frac{\partial L}{\partial \dot{q}}|_o + \epsilon^2\frac{\partial \tilde{L}}{\partial \dot{Q}} + \mathcal{O}(\epsilon^3)\Big) =: p^o + \epsilon\hat{P} + \epsilon^2 F + \mathcal{O}(\epsilon^3)\,, \tag{11}$$

where $p^o := \hat{p}(q^o,\dot{q}^o)$ are the momenta corresponding to the solution of the e.o.m. and $F(Q,\dot{Q})$ are functions quadratic in $Q,\dot{Q}$ that can be easily determined. Note that $\hat{P}$ define the Legendre map for the theory of linearized fluctuations. We see that at first order in ϵ the expansion for $\hat{p}$ behaves as expected.

The canonical Hamiltonian $H(q,p)$ associated with the Lagrangian L is characterized by $H(q,\hat{p}) = \hat{p}\dot{q} - L$. It was shown by Dirac that this function always exists and it is only determined up to the addition of primary Hamiltonian constraints ϕ_μ.

The fluctuation momenta P are defined in the canonical formalism from

$$p =: p^o + \epsilon P\,. \tag{12}$$

Comparing this expansion with the expansion (11), we find that the pullback map $p \to \hat{p}$ implies, under the change of variables (4) and (12), the map $P \to \hat{P} + \epsilon F(Q,\dot{Q}) + \mathcal{O}(\epsilon^2)$, that is,

$$p \to \hat{p} \quad \Rightarrow \quad P \to \hat{P} + \epsilon F(Q,\dot{Q}) + \mathcal{O}(\epsilon^2)\,. \tag{13}$$

which is different from the pullback map in the canonical fluctuations theory $P \to \hat{P}$. This mismatch between the two pullback operations —for the original theory and for the fluctuations theory— is of order ϵ and has no consequences as regards the mutual consistency of the Lagrangian and Hamiltonian version of the fluctuations theory.

Now consider the expansion for the primary Hamiltonian constraints,

$$\begin{aligned}\phi_\mu(q,p) &= \phi_\mu(q^o,p^o)+\epsilon(Q\frac{\partial\phi_\mu}{\partial q}|_o+P\frac{\partial\phi_\mu}{\partial p}|_o)+\epsilon^2 B_\mu(Q,P)+\mathcal{O}(\epsilon^3)\\ &=: \epsilon\tilde{\phi}_\mu(Q,P)+\epsilon^2 B_\mu(Q,P)+\mathcal{O}(\epsilon^3)\,,\end{aligned} \tag{14}$$

which, again, suggests that

$$\tilde{\phi}_\mu := Q\frac{\partial\phi_\mu}{\partial q}|_o+P\frac{\partial\phi_\mu}{\partial p}|_o \tag{15}$$

are the primary constraints for the canonical theory of linearized fluctuations. $B_\mu(Q,P)$ are functions quadratic in Q,P,

$$B_\mu(Q,P):=\frac{1}{2}\Big(Q\frac{\partial^2\phi_\mu}{\partial q\partial q}|_oQ+2Q\frac{\partial^2\phi_\mu}{\partial q\partial p}|_oP+P\frac{\partial^2\phi_\mu}{\partial p\partial p}|_oP\Big)\,. \tag{16}$$

4.1 Primary constraints

Let us verify that $\tilde{\phi}_\mu$ are indeed the primary Hamiltonian constraints for the theory originating in the fluctuations Lagrangian $\tilde{L}$. Since $\phi_\mu(q,\hat{p})=0$ identically[5], we also have

$$\frac{\partial\phi_\mu}{\partial q}|_o+\frac{\partial\phi_\mu}{\partial p}|_o\frac{\partial\hat{p}}{\partial q}|_o=0\,,$$

which implies

$$\tilde{\phi}_\mu(Q,P)=(P-Q\frac{\partial\hat{p}}{\partial q}|_o)\frac{\partial\phi_\mu}{\partial p}|_o=\Big((P-Q\frac{\partial L}{\partial q\partial\dot{q}})\gamma_\mu\Big)|_o\,.$$

Now one can check that $\tilde{\phi}_\mu(Q,\hat{P})=0$ identically. Indeed, $\hat{P}(Q,\dot{Q})=\frac{\partial\tilde{L}}{\partial\dot{Q}}$, and using (6),

$$\hat{P}(Q,\dot{Q})=Q\frac{\partial^2 L}{\partial q\partial\dot{q}}|_o+\dot{Q}W|_o\,.$$

Now, since $\gamma_\mu|_o$ are the null vectors of the Hessian matrix $W|_o$, we obtain that

$$\Big((\hat{P}-Q\frac{\partial L}{\partial q\partial\dot{q}})\gamma_\mu\Big)|_o=0$$

identically, which proves that indeed $\tilde{\phi}_\mu$ are the primary constraints for the canonical theory of linearized fluctuations.

4.2 The canonical Hamiltonian

Here we find the quadratic Hamiltonian for the linearized fluctuations. Consider the canonical Hamiltonian $H(q,p)$[6] and expand it in ϵ. This will define a candidate $\bar{H}(Q,P)$ for the quadratic

[5] Note that $\phi_\mu(q,p)\simeq 0$ (the constraints vanish on the constraint surface, but their derivatives not), while $\phi_\mu(q,\hat{p})=0$ identically.

[6] As said before, the canonical Hamiltonian is not unique, but any choice satisfying $H(q,\hat{p})=\hat{p}\dot{q}-L(q,\dot{q})$ will work.

canonical Hamiltonian of linearized fluctuations, but it should be checked that $\bar{H}(Q,\hat{P}) = \hat{P}\dot{Q} - \tilde{L}$. We will see that it is not exactly so. The reason is that, as Dirac already emphasized, the true Hamiltonian dynamics is described by the Dirac Hamiltonian

$$H_D(q,p) := H(q,p) + \lambda^\mu \phi_\mu \,. \tag{17}$$

In order to find the quadratic canonical Hamiltonian for fluctuations theory, let us first expand the canonical Hamiltonian,

$$H(q,p) =: H(q^o,p^o) + \epsilon(Q\frac{\partial H}{\partial q}|_o + P\frac{\partial H}{\partial p}|_o) + \epsilon^2 \bar{H}(Q,P) + \mathcal{O}(\epsilon^3)\,,$$

where $\bar{H}(Q,P)$ is quadratic in Q, P,

$$\bar{H}(Q,P) := \frac{1}{2}\Big(Q\frac{\partial^2 H}{\partial q \partial q}|_o Q + 2Q\frac{\partial^2 H}{\partial q \partial p}|_o P + P\frac{\partial^2 H}{\partial p \partial p}|_o P\Big)\,. \tag{18}$$

Since q^o, p^o satisfy the e.o.m.,

$$\begin{aligned} \dot{q}^o &= \frac{\partial H}{\partial p}|_o + \lambda^\mu(q^o,\dot{q}^o)\frac{\partial \phi_\mu}{\partial p}|_o \\ \dot{p}^o &= -\frac{\partial H}{\partial q}|_o - \lambda^\mu(q^o,\dot{q}^o)\frac{\partial \phi_\mu}{\partial q}|_o\,, \end{aligned} \tag{19}$$

where the Lagrange multipliers λ^μ can always be determined as definite functions in tangent space by using the e.o.m. for q and the pullback $p \to \hat{p}$ (see Appendix B for more details), we can replace

$$\begin{aligned} H(q,p) &= H(q^o,p^o) + \epsilon\Big(Q(-\dot{p}^o - \lambda^\mu(q^o,\dot{q}^o)\frac{\partial \phi_\mu}{\partial q}|_o) + P(\dot{q}^o - \lambda^\mu(q^o,\dot{q}^o)\frac{\partial \phi_\mu}{\partial p}|_o)\Big) \\ &+ \epsilon^2 \bar{H}(Q,P) + \mathcal{O}(\epsilon^3) \\ &= H(q^o,p^o) + \epsilon(P\dot{q}^o - Q\dot{p}^o) - \epsilon\lambda^\mu(q^o,\dot{q}^o)\tilde{\phi}_\mu + \epsilon^2\bar{H}(Q,P) + \mathcal{O}(\epsilon^3)\,, \end{aligned} \tag{20}$$

where the definition (15) has been used. Now consider the expansion for the functions $\lambda^\mu(q,\dot{q})$,

$$\lambda^\mu(q,\dot{q}) = \lambda^\mu(q^o,\dot{q}^o) + \epsilon\tilde{\lambda}^\mu(Q,\dot{Q}) + \mathcal{O}(\epsilon^2)\,,$$

and recall (14). Then, the Dirac Hamiltonian (17) has the expansion

$$\begin{aligned} H_D(q,p) &= H(q^o,p^o) + \epsilon(P\dot{q}^o - Q\dot{p}^o) \\ &+ \epsilon^2\Big(\bar{H}(Q,P) + \lambda^\mu(q^o,\dot{q}^o)B_\mu(Q,P) + \tilde{\lambda}^\mu\tilde{\phi}_\mu(Q,P)\Big) + \mathcal{O}(\epsilon^3)\,. \end{aligned} \tag{21}$$

Unlike in the expansion of L given by eq. (5), where the linear term does not contribute to the Lagrangian e.o.m., now the linear term in H_D does contribute to the Hamiltonian e.o.m. At the end of this subsection we will see that this is consistent with the Hamilton's equations of the original theory.

Expecting that the quadratic Dirac Hamiltonian of linearized fluctuations has the usual form $\tilde{H}_D(Q,P) = \tilde{H}(Q,P) + \tilde{\lambda}^\mu(Q,\dot{Q})\tilde{\phi}_\mu(Q,P)$, the result (21) strongly suggests that the true canonical Hamiltonian for the fluctuations is not $\bar{H}(Q,P)$, but the whole expression $\bar{H}(Q,P) + \lambda^\mu(q^o,\dot{q}^o)B_\mu(Q,P)$. We shall prove this assertion in the following. Equation (13) allows us to substitute $\hat{p}$ for p in the l.h.s. of (21), and $\hat{P} + \epsilon F + \mathcal{O}(\epsilon^2)$ for P in the r.h.s. Taking into account that $H_D(q,\hat{p}) = H(q,\hat{p})$ for $\phi(q,\hat{p}) = 0$, we obtain

$$\begin{aligned} H_D(q,\hat{p}) &= H(q^o,p^o) + \epsilon(\hat{P}\dot{q}^o - Q\dot{p}^o) \\ &+ \epsilon^2\Big(\bar{H}(Q,\hat{P}) + \lambda^\mu(q^o,\dot{q}^o)B_\mu(Q,\hat{P}) + \dot{q}^o F(Q,\dot{Q})\Big) + \mathcal{O}(\epsilon^3)\,. \end{aligned} \tag{22}$$

This expression must be compared with what we obtain directly from the fact that $H(q,\hat{p}) = \hat{p}\dot{q} - L$. Using (12) and (5)

$$\begin{aligned} H(q,\hat{p}) &= \hat{p}\dot{q} - L(q,\dot{q}) = \Big(p^o + \epsilon\hat{P} + \epsilon^2 F + \mathcal{O}(\epsilon^3)\Big)(\dot{q}^o + \epsilon\dot{Q}) \\ &- \Big(L(q^o,\dot{q}^o) + \epsilon\frac{d}{dt}(Qp^o) + \epsilon^2\tilde{L}(Q,\dot{Q}) + \mathcal{O}(\epsilon^3)\Big) \\ &= p^o\dot{q}^o - L(q^o,\dot{q}^o) + \epsilon(\hat{P}\dot{q}^o - Q\dot{p}^o) + \epsilon^2(\hat{P}\dot{Q} - \tilde{L} + \dot{q}^o F) + \mathcal{O}(\epsilon^3) \\ &=: H(q^o,p^o) + \epsilon(\hat{P}\dot{q}^o - Q\dot{p}^o) + \epsilon^2(\tilde{H}(Q,\hat{P}) + \dot{q}^o F) + \mathcal{O}(\epsilon^3)\,, \end{aligned} \tag{23}$$

where we have defined the true canonical Hamiltonian $\tilde{H}(Q,P)$ such that $\tilde{H}(Q,\hat{P}) = \hat{P}\dot{Q} - \tilde{L}$.

Now we compare (22) and (23). It follows that

$$\tilde{H}(Q,\hat{P}) = \bar{H}(Q,\hat{P}) + \lambda^\mu(q^o,\dot{q}^o)B_\mu(Q,\hat{P})\,,$$

and hence the canonical quadratic Hamiltonian for the fluctuations is

$$\tilde{H}(Q,P) = \bar{H}(Q,P) + \lambda^\mu(q^o,\dot{q}^o)B_\mu(Q,P)\,. \tag{24}$$

This proves our assertion.

Now (22) can be written as

$$H_D = H(q^o,p^o) + \epsilon(P\dot{q}^o - Q\dot{p}^o) + \epsilon^2\tilde{H}_D + \mathcal{O}(\epsilon^3)\,, \tag{25}$$

with $\tilde{H}_D = \tilde{H} + \tilde{\lambda}^\mu\tilde{\phi}_\mu\,.$

Now we can state the following result: *if $Q(t), P(t)$ is a solution of the Hamilton-Dirac's equations for the fluctuation dynamics, then $q(t) := q^o(t) + \epsilon Q(t)\,,\ \ p(t) := p^o(t) + \epsilon P(t)$ is a solution of the Hamilton's equations for the original dynamics up to terms of order ϵ^2.*

To prove it just consider the equations of the original dynamics and use (25),

$$\begin{aligned} \dot{q} &= \frac{1}{\epsilon}\frac{\partial H_D}{\partial P} = \frac{1}{\epsilon}\Big(\epsilon\dot{q}^o + \epsilon^2\frac{\partial\tilde{H}_D}{\partial P} + \mathcal{O}(\epsilon^3)\Big) = \dot{q}^o + \epsilon\frac{\partial\tilde{H}_D}{\partial P} + \mathcal{O}(\epsilon^2)\,, \\ \dot{p} &= -\frac{1}{\epsilon}\frac{\partial H_D}{\partial Q} = -\frac{1}{\epsilon}\Big(-\epsilon\dot{p}^o + \epsilon^2\frac{\partial\tilde{H}_D}{\partial Q} + \mathcal{O}(\epsilon^3)\Big) = \dot{p}^o - \epsilon\frac{\partial\tilde{H}_D}{\partial Q} + \mathcal{O}(\epsilon^2)\,. \end{aligned}$$

Since the equations for the fluctuation dynamics are

$$\dot{Q} = \frac{\partial\tilde{H}_D}{\partial P}\,, \qquad \dot{P} = -\frac{\partial\tilde{H}_D}{\partial Q}\,, \tag{26}$$

the result follows.

4.3 The algebra of constraints

The change of variables $q \to Q,\ p \to P$ is canonical up to a factor ϵ^2. We define a new bracket for the theory of fluctuations by

$$\{-,\,-\}\tilde{} = \epsilon^2\{-,\,-\}$$

in order to have $\{Q,\,P\}\tilde{} = \{q,\,p\} = \delta$.

Let us also define the auxiliary differential operator D^h, acting on functions of the original variables q, p, as $D^h := (Q\frac{\partial}{\partial q} + P\frac{\partial}{\partial p})|_o$, so that, for any $f(q, p; t)$, it gives the first order term in the expansion, $f = f|_o + \epsilon\, D^h f + \mathcal{O}(\epsilon^2)$, where $f|_o = f(q^o, p^o; t)$. Now consider the constraints $\tilde{\phi}_\mu$. Let us first evaluate their Poisson brackets. Using (65) we find

$$\{\tilde{\phi}_\mu,\, \tilde{\phi}_\nu\}\tilde{} = \{\phi_\mu,\, \phi_\nu\}|_o\,. \tag{27}$$

Thus the structure —First Class, Second Class— of the primary constraints is fully inherited in the fluctuations formalism and is fixed "on shell" . Suppose that the original primary constraints $\phi_\mu \equiv \phi_\mu^{(0)}$ (the superindex ${}^{(0)}$ is for primary) split into First Class constraints $\phi_{\mu_0}^{(0)}$ and Second Class constraints $\phi_{\mu_0'}^{(0)}$. Then the secondary constraints are obtained as $\phi_{\mu_0}^{(1)} := \{\phi_{\mu_0}^{(0)},\, H\}$. Now we proceed in the same way with the fluctuations theory. The same splitting repeats for the constraints $\tilde{\phi}_\mu^{(0)}$. The only difference in finding the secondary constraints is that $\tilde{\phi}_\mu^{(0)}$ are in general time dependent.

Then, using the form of the Hamiltonian (24) and the fact that it is always $\tilde{\phi} = D^h\phi$ (for any indices), for the secondary constraints of the fluctuations theory we obtain

$$\begin{aligned}
\tilde{\phi}_{\mu_0}^{(1)} &:= \frac{\partial}{\partial t}\tilde{\phi}_{\mu_0}^{(0)} + \{\tilde{\phi}_{\mu_0}^{(0)},\, \tilde{H}\}\tilde{} = Q\frac{d}{dt}\frac{\partial\phi_{\mu_0}^{(0)}}{\partial q}|_o + P\frac{d}{dt}\frac{\partial\phi_{\mu_0}^{(0)}}{\partial p}|_o + \{\tilde{\phi}_{\mu_0}^{(0)},\, \bar{H}\}\tilde{} \\
&+ \lambda^\mu(q^o, \dot{q}^o)\{\tilde{\phi}_{\mu_0}^{(0)},\, B_\mu\}\tilde{}\,.
\end{aligned} \tag{28}$$

The bracket in the last term can be transformed with the help of the expansion (14) and the identity (66) (see Appendix A)

$$\{\tilde{\phi}_{\mu_0}^{(0)},\, B_\mu\}\tilde{} = \{\tilde{\phi}_\mu^{(0)},\, B_{\mu_0}\}\tilde{} + D^h\{\phi_{\mu_0}^{(0)},\, \phi_\mu^{(0)}\}\,. \tag{29}$$

Since $\phi_{\mu_0}^{(0)}$ are First Class constraints, $\{\phi_{\mu_0}^{(0)},\, \phi_\mu^{(0)}\} = \alpha_{\mu_0\mu}^\nu\phi_\nu^{(0)}$ for some functions $\alpha_{\mu_0\mu}^\nu$. Therefore,

$$D^h\{\phi_{\mu_0}^{(0)},\, \phi_\mu^{(0)}\} = D^h(\alpha_{\mu_0\mu}^\nu\phi_\nu^{(0)}) = \alpha_{\mu_0\mu}^\nu|_o D^h\phi_\nu^{(0)} = \alpha_{\mu_0\mu}^\nu|_o\tilde{\phi}_\nu^{(0)} \simeq 0\,, \tag{30}$$

which shows that the last term in (29) vanishes on the surface of primary constraints (this is the meaning of $\simeq$ at this stage).

Using the previous results and the e.o.m. (19), a little computation shows that

$$\begin{aligned}
& Q\frac{d}{dt}\frac{\partial\phi_{\mu_0}^{(0)}}{\partial q}|_o + P\frac{d}{dt}\frac{\partial\phi_{\mu_0}^{(0)}}{\partial p}|_o + \lambda^\mu(q^o, \dot{q}^o)\{\tilde{\phi}_{\mu_0}^{(0)},\, B_\mu\}\tilde{} \\
\simeq\;& Q\frac{d}{dt}\frac{\partial\phi_{\mu_0}^{(0)}}{\partial q}|_o + P\frac{d}{dt}\frac{\partial\phi_{\mu_0}^{(0)}}{\partial p}|_o + \lambda^\mu(q^o, \dot{q}^o)\{\tilde{\phi}_\mu^{(0)},\, B_{\mu_0}\}\tilde{} \\
=\;& Q\{\frac{\partial\phi_{\mu_0}^{(0)}}{\partial q},\, H\}|_o + P\{\frac{\partial\phi_{\mu_0}^{(0)}}{\partial p},\, H\}|_o\,,
\end{aligned} \tag{31}$$

whereas

$$\{\tilde{\phi}^{(0)}_{\mu_0},\, \bar{H}\}\tilde{} = Q\{\phi^{(0)}_{\mu_0},\, \frac{\partial H}{\partial q}\}|_o + P\{\phi^{(0)}_{\mu_0},\, \frac{\partial H}{\partial p}\}|_o\,. \tag{32}$$

Altogether gives

$$\tilde{\phi}^{(1)}_{\mu} \simeq (Q\frac{\partial}{\partial q} + P\frac{\partial}{\partial p})\{\phi^{(0)}_{\mu_0},\, H\}|_o = (Q\frac{\partial}{\partial q} + P\frac{\partial}{\partial p})\phi^{(1)}_{\mu_0}|_o = D^h\phi^{(1)}_{\mu_0}\,,$$

Note the complete analogy of the expansion of secondary constraints with that of the primary constraints (14). Namely, there we had the expansion

$$\phi^{(0)}_{\mu}(q,p) = \epsilon\tilde{\phi}^{(0)}_{\mu}(Q,P) + \mathcal{O}(\epsilon^2),$$

now we find

$$\phi^{(1)}_{\mu_0}(q,p) \simeq \epsilon\tilde{\phi}^{(1)}_{\mu_0}(Q,P) + \mathcal{O}(\epsilon^2)\,. \tag{33}$$

On the other hand, recalling Appendix A,

$$\{D^h f,\, D^h g\}\tilde{} = \{f,\, g\}|_o\,. \tag{34}$$

Thus *our algebra of primary and secondary constraints for the fluctuations theory just mimics the algebra of the original constraints computed at* q^o, p^o. This means in particular that the First Class constraints become Abelianized for the fluctuations theory.

In Sec.3 we showed that the original and fluctuations Lagrangians have equivalent dynamical structures in the case of quadratic Lagrangians. Thus, the fact that the fluctuations Lagrangian cannot have non-Abelian symmetry means that non-Abelian symmetry cannot be described by a quadratic Lagrangian.

4.4 Dirac brackets

The constraint algorithm now continues in parallel for the original theory and for the fluctuations theory. Let us relate the Dirac brackets at the level of the primary constraints for both theories. The matrix of Second Class constraints

$$\{\tilde{\phi}^{(0)}_{\mu'_0},\, \tilde{\phi}^{(0)}_{\nu'_0}\}\tilde{} = \{D^h\phi^{(0)}_{\mu'_0},\, D^h\phi^{(0)}_{\nu'_0}\}\tilde{} = \{\phi^{(0)}_{\mu'_0},\, \phi^{(0)}_{\nu'_0}\}|_o =: M_{\mu'_0\nu'_0}|_o\,,$$

is invertible. Consider the inverse $M^{\mu'_0\nu'_0}|_o$. Dirac brackets for the fluctuations theory are then defined by

$$\{-\,,-\}^{\tilde{*}} := \{-\,,-\}\tilde{} - \{-\,,\tilde{\phi}^{(0)}_{\mu'_0}\}\tilde{}\, M^{\mu'_0\nu'_0}|_o\,\{\tilde{\phi}^{(0)}_{\nu'_0}\,,-\}\tilde{}$$

Then, for any functions f, g in the original phase space, the following Dirac bracket can be calculated,

$$\begin{aligned}\{D^h f\,, D^h g\}^{\tilde{*}} &:= \{D^h f\,, D^h g\}\tilde{} - \{D^h f\,, \tilde{\phi}^{(0)}_{\mu'_0}\}\tilde{}\, M^{\mu'_0\nu'_0}|_o\,\{\tilde{\phi}^{(0)}_{\nu'_0}\,, D^h g\}\tilde{}\\ &= \{f\,,g\}|_o - \{f\,\phi^{(0)}_{\mu'_0}\}|_o\, M^{\mu'_0\nu'_0}|_o\,\{\phi^{(0)}_{\nu'_0}\,,g\}|_o = \{f\,,g\}^*|_o\,.\end{aligned} \tag{35}$$

Equation (35) is the analogous of (34), now for Dirac brackets.

The knowledge that the primary constraints $\tilde{\phi}^{(0)}_{\mu}$ and the secondary constraints $\tilde{\phi}^{(1)}_{\mu_0}$ for the fluctuations theory are just $\tilde{\phi}_\mu = D^h\phi_\mu$ and $\tilde{\phi}^{(1)}_{\mu_0} = D^h\phi^{(1)}_{\mu_0}$, together with the results (34) and (35), allows to continue the constraint algorithm another step. The same parallelisms continue until the algorithm is finished. This proves that *the full algebra of constraints in the fluctuations theory mimics the algebra of the original constraints computed at* q^o, p^o*. In consequence, the original theory and the fluctuations theory have the same number of physical degrees of freedom.* The Abelianization of the First Class constraints for the fluctuations theory is a general phenomenon[7]. Since combinations of the First Class constraints generate gauge symmetries, and since their number remains unchanged, the dimensions of the original gauge group and the Abelian gauge group in the fluctuations theory, are the same.

4.5 Connection with the Lagrangian constraints

Using results in [9], the primary Lagragian constraints can be written as

$$\chi_{\mu_0} = \mathcal{F}L^*\{\phi^{(0)}_{\mu_0}, H\} = \mathcal{F}L^*\phi^{(1)}_{\mu_0} \tag{36}$$

$$\chi_{\mu'_0} = \mathcal{F}L^*\{\phi^{(0)}_{\mu'_0}, H\} + \lambda^{\nu'_0}(q,\dot{q})\mathcal{F}L^*\{\phi^{(0)}_{\mu'_0}, \phi^{(0)}_{\nu'_0}\}\,. \tag{37}$$

The rationale of (36) is that the pullback of a Hamiltonian constraint must be a Lagrangian constraint. As for (37) the idea is that the time evolution of a —now Second Class— constraint must also vanish "on shell" (here it is relevant that the Lagrange multipliers $\lambda^{\nu'_0}$ are definite functions in tangent space).

Let us use the notation[8] $D^l := (Q\frac{\partial}{\partial q} + \dot{Q}\frac{\partial}{\partial \dot{q}})|_o$ and the fact, easily proved, that

$$D^l \circ \mathcal{F}L^* = \tilde{\mathcal{F}L}^* \circ D^h\,,$$

where $\tilde{\mathcal{F}L}^*$ is the pullback operation $P \to \hat{P}$ for the fluctuations theory. Now expand (36) in ϵ, we get

$$\chi_{\mu_0} = \epsilon D^l(\mathcal{F}L^*\phi^{(1)}_{\mu_0}) + \mathcal{O}(\epsilon^2) = \epsilon\tilde{\mathcal{F}L}^*(D^h\phi^{(1)}_{\mu_0}) + \mathcal{O}(\epsilon^2) = \epsilon\tilde{\mathcal{F}L}^*\tilde{\phi}^{(1)}_{\mu_0} + \mathcal{O}(\epsilon^2)\,, \tag{38}$$

which means that indeed the relation (36) is preserved for the fluctuations theory as well, that is, $\tilde{\chi}_{\mu_0} = \tilde{\mathcal{F}L}^*\tilde{\phi}^{(1)}_{\mu_0}$.

Let us do the same with (37),

$$\begin{aligned}\chi_{\mu'_0} &= \epsilon\Big(D^l(\mathcal{F}L^*\{\phi^{(0)}_{\mu'_0}, H\}) + D^l(\lambda^{\nu'_0}\mathcal{F}L^*\{\phi^{(0)}_{\mu'_0}, \phi^{(0)}_{\nu'_0}\})\Big) + \mathcal{O}(\epsilon^2)\\ &= \epsilon\Big(\tilde{\mathcal{F}L}^*(D^h\{\phi^{(0)}_{\mu'_0}, H\}) + (D^l\lambda^{\nu'_0})\{\phi^{(0)}_{\mu'_0}, \phi^{(0)}_{\nu'_0}\}|_o + \lambda^{\nu'_0}|_o\tilde{\mathcal{F}L}^*(D^h\{\phi^{(0)}_{\mu'_0}, \phi^{(0)}_{\nu'_0}\})\Big) + \mathcal{O}(\epsilon^2)\\ &= \epsilon\Big(\tilde{\mathcal{F}L}^*(D^h\{\phi^{(0)}_{\mu'_0}, H + \lambda^{\nu'_0}|_o\phi^{(0)}_{\nu'_0}\}) + (D^l\lambda^{\nu'_0})\{\phi^{(0)}_{\mu'_0}, \phi^{(0)}_{\nu'_0}\}|_o\Big) + \mathcal{O}(\epsilon^2)\,. \end{aligned}\tag{39}$$

[7] In fact gauge symmetries for quadratic systems are always Abelian because the Hamiltonian constraints are linear –thus their Poisson bracket is field independent– and hence the only way to exhibit First Class constraints is through the vanishing of their Poisson brackets with all the constraints. Non-Abelian theories and self-interaction –associated with terms in the action of order higher than quadratic– go hand in hand.

[8] D^l plays the same role in tangent space as D^h plays in phase space.

Recalling that $D^l\lambda^{\nu'_0} = \tilde{\lambda}^{\nu'_0}$, the form of the expansion (39) indicates that the objects

$$\tilde{\mathcal{F}L}^*(D^h\{\phi^{(0)}_{\mu'_0},\, H + \lambda^{\nu'_0}|_o\phi^{(0)}_{\nu'_0}\}) + \tilde{\lambda}^{\nu'_0}\{\phi^{(0)}_{\mu'_0},\, \phi^{(0)}_{\nu'_0}\}|_o\,, \tag{40}$$

should be the primary Lagrangian constraints $\tilde{\chi}_{\mu'_0}$ for the fluctuations theory. Let us check that this is indeed the case. Working only with the theory of fluctuations and using (29) and the analogous of (31), (32), now applied to $\tilde{\phi}^{(0)}_{\mu'_0}$, we get

$$\begin{aligned}
\tilde{\chi}_{\mu'_0} &= \tilde{\mathcal{F}L}^*\Big(\frac{\partial}{\partial t}\tilde{\phi}^{(0)}_{\mu'_0} + \{\tilde{\phi}^{(0)}_{\mu'_0},\, \tilde{H}\}\tilde{}\Big) + \tilde{\lambda}^{\nu'_0}\tilde{\mathcal{F}L}^*\{\tilde{\phi}^{(0)}_{\mu'_0},\, \tilde{\phi}^{(0)}_{\nu'_0}\}\tilde{} \\
&= \tilde{\mathcal{F}L}^*\Big(\frac{\partial}{\partial t}\tilde{\phi}^{(0)}_{\mu'_0} + \lambda^{\mu}|_o\{\tilde{\phi}^{(0)}_{\mu'_0},\, B_\mu\}\tilde{}\Big) + \tilde{\mathcal{F}L}^*\{\tilde{\phi}^{(0)}_{\mu'_0},\, \bar{H}\}\tilde{} + \tilde{\lambda}^{\nu'_0}\tilde{\mathcal{F}L}^*\{\tilde{\phi}^{(0)}_{\mu'_0},\, \tilde{\phi}^{(0)}_{\nu'_0}\}\tilde{} \\
&= \tilde{\mathcal{F}L}^*\Big(\frac{\partial}{\partial t}\tilde{\phi}^{(0)}_{\mu'_0} + \lambda^{\mu}|_o(\{\tilde{\phi}^{(0)}_{\mu},\, B_{\mu'_0}\}\tilde{} + D^h\{\phi^{(0)}_{\mu'_0},\, \phi^{(0)}_{\mu}\})\Big) + \tilde{\mathcal{F}L}^*\{\tilde{\phi}^{(0)}_{\mu'_0},\, \bar{H}\}\tilde{} + \tilde{\lambda}^{\nu'_0}\tilde{\mathcal{F}L}^*\{\tilde{\phi}^{(0)}_{\mu'_0},\, \tilde{\phi}^{(0)}_{\nu'_0}\}\tilde{} \\
&= \tilde{\mathcal{F}L}^*(D^h\{\phi^{(0)}_{\mu'_0},\, H\}) + \lambda^{\mu}|_o\tilde{\mathcal{F}L}^*(D^h\{\phi^{(0)}_{\mu'_0},\, \phi^{(0)}_{\mu}\}) + \tilde{\lambda}^{\nu'_0}\{\phi^{(0)}_{\mu'_0},\, \tilde{\phi}^{(0)}_{\nu'_0}\}|_o \\
&= \tilde{\mathcal{F}L}^*(D^h\{\phi^{(0)}_{\mu'_0},\, H\}) + \lambda^{\nu'_0}|_o\tilde{\mathcal{F}L}^*(D^h\{\phi^{(0)}_{\mu'_0},\, \phi^{(0)}_{\nu'_0}\}) + \tilde{\lambda}^{\nu'_0}\{\phi^{(0)}_{\mu'_0},\, \tilde{\phi}^{(0)}_{\nu'_0}\}|_o \\
&= \tilde{\mathcal{F}L}^*(D^h\{\phi^{(0)}_{\mu'_0},\, H + \lambda^{\nu'_0}|_o\phi^{(0)}_{\nu'_0}\}) + \tilde{\lambda}^{\nu'_0}\{\phi^{(0)}_{\mu'_0},\, \phi^{(0)}_{\nu'_0}\}|_o\,,
\end{aligned} \tag{41}$$

which is exactly (40).

The constraint algorithm in tangent space for the fluctuations theory continues in the same way. The result is that for each constraint χ of the original theory there is a constraint $\tilde{\chi}$ in the fluctuations theory, that can be determined through the expansion $\chi = \epsilon\tilde{\chi} + \mathcal{O}(\epsilon^2)$.

5 Noether symmetries

Noether symmetries of the action are those continuous symmetries for which the infinitesimal transformation δq induced on the Lagrangian L gives a total derivative –a divergence in field theory. They exhibit a conserved quantity –conserved current in field theory– G such that the equality

$$[L]_q\delta q + \frac{d}{dt}G = 0 \tag{42}$$

holds identically. Let us ϵ-expand (42) according to (4), using the expansion (7) which includes the next order in ϵ. Note that $\delta q = \delta q|_o + \epsilon D^l\delta q + \mathcal{O}(\epsilon^2)$, and $G = G|_o + \epsilon D^l G + \epsilon^2 D^{2l}G + \mathcal{O}(\epsilon^3)$, where we have introduced the notation $D^{2l}f$ for the second order term in the expansion of any $f(q, \dot{q}, t)$,

$$D^{2l}f = \frac{1}{2}\Big(Q\frac{\partial^2 f}{\partial q\partial q}|_o Q + 2Q\frac{\partial^2 f}{\partial q\partial \dot{q}}|_o\dot{Q} + \dot{Q}\frac{\partial^2 f}{\partial \dot{q}\partial \dot{q}}|_o\dot{Q}\Big)\,. \tag{43}$$

Now the l.h.s. of (42) becomes

$$\begin{aligned}
[L]_q\delta q + \frac{d}{dt}G &= \Big(\epsilon[\tilde{L}(Q,\dot{Q})]_Q + \epsilon^2[D^{2l}L]_Q + \mathcal{O}(\epsilon^3)\Big)\Big(\delta q|_o + \epsilon D^l\delta q + \mathcal{O}(\epsilon^2)\Big) \\
&+ \frac{d}{dt}\Big(G|_o + \epsilon D^l G + \epsilon^2 D^{2l}G + \mathcal{O}(\epsilon^3)\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \epsilon\Big([\tilde{L}(Q,\dot{Q})]_Q \delta q|_o + \frac{d}{dt}D^l G\Big) \\
&+ \epsilon^2\Big([\tilde{L}(Q,\dot{Q})]_Q D^l\delta q + [D^{2l}L]_Q\delta q|_o + \frac{d}{dt}(D^{2l}G)\Big)\,,
\end{aligned}
\tag{44}
$$

where $\frac{d}{dt}G|_o$ vanishes because $G|_o$ is a conserved quantity evaluated on a solution. Thus (42) implies, to the lowest orders in the expansion,

$$
[\tilde{L}(Q,\dot{Q})]_Q \delta q|_o + \frac{d}{dt}D^l G = 0 \tag{45}
$$

and

$$
[\tilde{L}(Q,\dot{Q})]_Q D^l\delta q + [D^{2l}L]_Q\delta q|_o + \frac{d}{dt}(D^{2l}G) = 0\,. \tag{46}
$$

According to (45), the transformations δQ defined by

$$
\delta Q := \delta q|_o \tag{47}
$$

produce a Noether symmetry for $\tilde{L}$, with a linear conserved quantity $D^l G$. Note that this symmetry is trivial when the original symmetry preserves the background, that is, when $\delta q|_o = 0$. However, in this case, the equation (46) takes the form of the conservation law (42). Indeed, when $\delta q|_o = 0$, from eq. (46), the transformations $\tilde{\delta} Q$ defined by

$$
\tilde{\delta} Q := D^l \delta q \tag{48}
$$

lead to a Noether symmetry with a quadratic conserved quantity $D^{2l}G$.

Equations (45) and (47), on one side, and (46) and (48), on the other, are the two standard mechanisms for which a Noether symmetry of L is inherited by $\tilde{L}$. We will call the corresponding conserved quantities *linear generators* and *quadratic generators*, respectively.[9] Let us make some comments on these two mechanisms.

(i) Summarizing the main result of this section, we observe that the presence of the classical solution –the background– causes a splitting of the original Noether symmetries according to whether they preserve the background, or are broken by it. Those that are broken by the background, equation (47) (first mechanism), will yield symmetries for the quadratic fluctuations action with *linear* generators. Instead, the symmetries that preserve the background, equation (48) (second mechanism), will yield symmetries with *quadratic* generators.

(ii) Note that gauge symmetries for the fluctuations theory can only be realized through the first mechanism (47). In fact, since gauge symmetries are generated in phase space by

[9] The terminology of *generators* stems from the canonical framework, and the action of these generators is produced by way of the Poisson bracket. There is the subtle point, however, that there may exist Noether symmetries in tangent space that can not be brought –i.e., projected– to phase space. In such case the connection of the infinitesimal transformation with the conserved quantity needs more elaboration (see [15]).

appropriate combinations of First Class constraints ϕ_i, the correspondence between First Class constraints of both theories already indicates that if

$$\eta\phi_1 + \dot{\eta}\phi_2 + ...$$

is a generator of Noether symmetries (with the gauge parameter $\eta(t)$) for the original theory, then

$$\eta\tilde{\phi}_1 + \dot{\eta}\tilde{\phi}_2 + ...$$

with $\tilde{\phi}_i = D^l\phi_i$, is a generator of gauge Noether symmetries for the fluctuations theory, with the additional fact, already pointed out, that these symmetries of $\tilde{L}$ are Abelian.

Noticing that the constraints of the fluctuations theory are linear, we infer that the gauge transformations for $\tilde{L}$ must be realized exclusively through the first mechanism (47). A somewhat unexpected consequence of this fact is the general result that gauge symmetries in the original theory that completely preserve the background cannot exist, otherwise the fluctuations theory would change the number of physical degrees of freedom. Only for particular restrictions on the gauge parameters, the gauge symmetries may preserve the background. Generally covariant theories –having solutions which may exhibit some Killing symmetries– and Yang-Mills gauge theories are obvious verifications of this assertion.

(*iii*) Since the symmetries provided by (46), that is, the Noether symmetries of $\tilde{L}$ inherited from the background-preserving symmetries of L, are always rigid, they do not change the number of physical degrees of freedom of the fluctuations theory. Note that they do not exist around any solution q^o, but only around particular backgrounds, for which $\delta q|_o = 0$.

(*iv*) We can keep looking at even higher orders of G in ϵ, generating, for instance, the transformation law $\tilde{\tilde{\delta}} Q = D^{2l}\delta q$. These transformations will emerge as rigid symmetries of $\tilde{L}$ when both $\delta q|_o$ and $D^l\delta q$ vanish. Therefore, the more non-linear the original theory is, the richer the structure of inherited rigid symmetries in the fluctuations theory is likely to be.

(*v*) In this section Lagrangian Noether symmetries have been studied. We can proceed in a similar way with a Hamiltonian generator G^H and, starting from (12), find the corresponding linear and quadratic generators in the canonical fluctuations theory. For transformations projectable to phase space, and belonging to the type that breaks the background, it can be shown that if G^H satisfies the conditions (spelled out in [16]) to be a generator of canonical Noether symmetries for the original theory, then D^hG^H satisfies the same conditions for the fluctuations theory and becomes a generator of canonical Noether symmetries for it.

(*vi*) The obstruction to the projectability of Noether transformations from tangent space to phase space [15] is related to the existence of a non-Abelian structure for the –primary and secondary– First Class constraints of the theory. In consequence, the symmetries of $\tilde{L}$ produced

by the first mechanism (45) are always projectable to phase space. In fact, they are field independent transformations.

(*vii*) One can easily verify that the projectability to phase space of the rigid Noether symmetries provided by (46) is directly related to the projectablility of the original transformation. The requirement of projectability of a given transformation δq is, using (3),

$$\gamma^i_\mu \frac{\partial}{\partial \dot{q}^i}\delta q^j = 0\,,$$

and for the transformations $\tilde{\delta} Q^j = D^l \delta q^j$ the requirement is, accordingly,

$$\gamma^i_\mu|_o \frac{\partial}{\partial \dot{Q}^i}\delta Q^j = 0\,,$$

where the zero modes of Hessian for $\tilde{L}$ are just $\gamma^i_\mu|_o$. However, considering that

$$\frac{\partial}{\partial \dot{Q}^i}\delta Q^j = \frac{\partial}{\partial \dot{Q}^i} D^l \delta q^j = (\frac{\partial}{\partial \dot{q}^i}\delta q^j)|_o\,,$$

we can infer

$$\gamma^i_\mu \frac{\partial}{\partial \dot{q}^i}\delta q^j = 0 \quad \Rightarrow (\gamma^i_\mu \frac{\partial}{\partial \dot{q}^i}\delta q^j)|_o = 0 \quad \Rightarrow \quad \gamma^i_\mu|_o \frac{\partial}{\partial \dot{Q}^i}\delta Q^j = 0\,,$$

which proves our assertion.

6 Examples

6.1 Massive relativistic free particle

Consider the Lagrangian of a massive free particle in Minkowski spacetime,

$$L = -m\sqrt{-\dot{\mathbf{x}}^2}\,, \tag{49}$$

with $\dot{\mathbf{x}}^2 = \eta_{\mu\nu}\,\dot{x}^\mu \dot{x}^\nu$ and $\dot{x}^\mu = \frac{dx^\mu}{d\tau}$, where $\eta_{\mu\nu}$ is the Minkowski metric with "mostly plus" signature. There are no Lagrangian constraints even though the Hessian $W_{\mu\nu} = \frac{m}{\sqrt{-\dot{\mathbf{x}}^2}}\,(\eta_{\mu\nu} - \frac{\dot{x}_\mu \dot{x}_\nu}{\dot{\mathbf{x}}^2})$ has one zero mode $\dot{x}^\nu$ ($W_{\mu\nu}\,\dot{x}^\nu = 0$), since the Euler-Lagrange e.o.m. are $[L]_\mu = -W_{\mu\nu}\,\ddot{x}^\nu$ and the expression $[L]_\mu \dot{x}^\nu$ vanishes identically.

The Lagrangian has the gauge symmetry of τ-reparameterizations $\delta\tau = \varepsilon$, under which the coordinates transform as $\delta\mathbf{x} = \varepsilon\,\dot{\mathbf{x}}$ and the Lagrangian as $\delta L = -\frac{d}{d\tau}\,(\varepsilon L)$. The momentum vector (indices are raised and lowered with $\eta_{\mu\nu}$)

$$\hat{\mathbf{p}} = \frac{\partial L}{\partial \dot{\mathbf{x}}} = m\,\frac{\dot{\mathbf{x}}}{\sqrt{-\dot{\mathbf{x}}^2}}\,, \tag{50}$$

satisfies $\hat{\mathbf{p}}^2 + m^2 = 0$ identically, thus showing the existence of a constraint in phase space

$$\phi = \frac{1}{2}\,(\mathbf{p}^2 + m^2) \simeq 0\,. \tag{51}$$

The canonical Hamiltonian vanishes because the Lagrangian is a degree one homogeneous function of the velocities. The Dirac Hamiltonian is therefore

$$H_D = \lambda\phi\,.$$

The arbitrary function in phase space λ is determined in tangent space by using the Hamiltonian e.o.m.

$$\dot{\mathbf{x}} = \{\mathbf{x},\, H_D\} = \lambda\,\{\mathbf{x},\, \phi\} = \lambda\mathbf{p}\,,$$

and applying the pullback map $p \to \hat{p}$ as defined in (50). We get

$$\lambda = \frac{\sqrt{-\dot{\mathbf{x}}^2}}{m}\,.$$

ϕ is the only constraint in phase space, and (being the First Class) it generates the gauge transformations

$$\begin{aligned}\delta\mathbf{x} &= \{\mathbf{x}, \alpha\phi\} = \alpha\mathbf{p}\,,\\ \delta\mathbf{p} &= \{\mathbf{p}, \alpha\phi\} = \mathbf{0}\,,\end{aligned}$$

with $\alpha(\tau)$ an arbitrary infinitesimal function. These transformations are τ-reparametrizations, and they can be put in the standard Lagrangian form for reparametrization invariant theories, $\delta\mathbf{x} = \varepsilon\dot{\mathbf{x}}$, after applying the pullback (50) and redefining the gauge parameter $\alpha(\tau) = \frac{\varepsilon(\tau)}{m}\sqrt{-\dot{\mathbf{x}}^2}$.

Now we will examine the fluctuations theory around a general solution of the e.o.m. of L. This solution has the form $\mathbf{x}^o = \mathbf{u}\,s(\tau) + \mathbf{c}$, with $\mathbf{u}, \mathbf{c}$ constant vectors. We can conventionally assume that $\mathbf{u}^2 = -1$. $s(\tau)$ is an arbitrary monotonically increasing function, $\dot{s}(\tau) =: v(\tau) > 0$. The fluctuations Lagrangian becomes

$$\tilde{L} = \frac{m}{2\,v(\tau)}\,\dot{\mathbf{Q}}\boldsymbol{\Gamma}\dot{\mathbf{Q}}\,,$$

where $\boldsymbol{\Gamma}$ is the projector transversal to the $\mathbf{u}$ direction, with the components $\Gamma_{\mu\nu} = \eta_{\mu\nu} + u_\mu u_\nu\,.$

The canonical Hamiltonian for fluctuations is, according to (24), $\tilde{H}(Q,P) = \bar{H}(Q,P) + \lambda(q^o,\dot{q}^o)B(Q,P)$, where here $\bar{H}(Q,P) = 0$ and $\lambda(q^o,\dot{q}^o) = \frac{v(\tau)}{m}$. The term $B(Q,P)$ in (14) is now $B(Q,P) = \frac{1}{2}\,\mathbf{P}^2$, and we obtain

$$\tilde{H}(Q,P) = \frac{v(\tau)}{2m}\,\mathbf{P}^2\,.$$

One can check that indeed $\tilde{H}(Q,\hat{P}) = \hat{\mathbf{P}}\mathbf{Q} - \tilde{L}(Q,\dot{Q})$.

6.1.1 Noether symmetries

Now we will discuss separately the different symmetries inherited in the fluctuations theory.

Gauge symmetries. The phase space constraint $\tilde{\phi}$ (recall that ϕ expands as $\phi = \epsilon\tilde{\phi} + \mathcal{O}(\epsilon^2)$) becomes

$$\tilde{\phi} = m(\mathbf{Pu})\,,$$

and it generates the gauge transformations $\delta\mathbf{Q} = \{\mathbf{Q}, \alpha\tilde{\phi}\}^{\tilde{}} = \alpha m\mathbf{u} = \alpha\mathbf{p^o}$ given by equation (47).

Rigid symmetries with linear generators. Let us see how the Poincaré symmetries of (49) appear in the fluctuations theory. The Poincaré transformations $\delta x^\mu = a^\mu + \omega^\mu{}_\nu x^\nu$, generated by $G_p = p_\mu\,(a^\mu + \omega^\mu{}_\nu x^\nu)$ (the subindex p is for Poincaré), become

$$\begin{aligned} \delta Q^\mu &= \delta x^\mu|_o = a^\mu + \omega^\mu{}_\nu(s(\tau)\,u^\nu + c^\nu) = a^\mu + \omega^\mu{}_\nu c^\nu + s(\tau)\omega^\mu{}_\nu u^\nu \\ &=: d^\mu + s(\tau)\,r^\mu\,, \end{aligned} \tag{52}$$

with d^μ arbitrary and r^μ satisfying $\mathbf{ru} = 0$. Both d^μ and r^μ infinitesimal vectors. Since the symmetry is Abelian and field-independent, the finite transformations have just the same form with d^μ and r^μ finite. The generator of these symmetries is just $D^hG_p = (d^\mu + s(\tau)r^\mu)P_\mu + mr_\mu Q^\mu$. Thus, as Noether symmetries for $\tilde{L}$ we obtain the usual translations $\mathbf{d}$ and the particular time dependent translations $s(\tau)\,\mathbf{r}$ orthogonal to $\mathbf{u}$.

The background is preserved for parameters $\omega^\mu{}_\nu$ and a^μ such that $\omega^\mu{}_\nu = 0$ and $a^\mu + \omega^\mu{}_\nu c^\nu = 0$. For this specific set of parameters, which imply $d^\mu = r^\mu = 0$ and hence $D^hG_p = 0$, the Poincaré symmetries will be realized with quadratic generators, as we show later.

Other rigid symmetries with linear generators. Besides the gauge and Poincaré symmetries, the free particle Lagrangian in Minkowski spacetime exhibits other Noether symmetries. Take for instance the quantity

$$m\,\mathbf{x}\,\mathbf{\Gamma}^{(\mathbf{p})}\mathbf{w},$$

with $\mathbf{\Gamma}^{(\mathbf{p})}$ being the projector transversal to the momentum $\mathbf{p}$,

$$\Gamma^{(\mathbf{p})}_{\mu\nu} = \eta_{\mu\nu} + \frac{p_\mu p_\nu}{-\mathbf{p}^2}\,,$$

and with $\mathbf{w}$ being an arbitrary –infinitesimal– constant vector. Since this quantity has no explicit time dependence and has vanishing Poisson bracket with the only constraint, ϕ, of the theory, it fulfills the conditions[10] $\frac{\partial G}{\partial t} + \{G,\,H\} \simeq \phi$ and $\{G,\,\phi\} \simeq \phi$ (recall that the canonical Hamiltonian vanishes in our case), to be a generator of canonical Noether transformations. In fact we can use a simpler version for it,

$$G_g = mx^\mu(\eta_{\mu\nu} + \frac{p_\mu p_\nu}{m^2})w^\nu\,,$$

where we have used the constraint ϕ. Now

$$\{G_g,\,\phi\} = \frac{2(\mathbf{wp})}{m}\,\phi\,,$$

[10]Here the notation "$\simeq$" means an equality on the primary constraint surface only.

which still fulfills the conditions for being a canonical generator. This generator produces the transformations

$$\delta\mathbf{x} = \{\mathbf{x},\, G_g\} = \frac{1}{m}\Big((\mathbf{pw})\mathbf{x} + (\mathbf{xp})\mathbf{w}\Big)\,,$$

which, restricted to the background $\mathbf{x}^o = \mathbf{u}s(\tau) + \mathbf{c}$, $\mathbf{p}^o = m\mathbf{u}$, give the transformations of the fluctuations

$$\delta\mathbf{Q} = \delta\mathbf{x}|_o = s(\tau)\Big(-\mathbf{w} + (\mathbf{uw})\mathbf{u}\Big) + (\mathbf{uc})\mathbf{w} + (\mathbf{uw})\mathbf{c}\,.$$

(Note that $\delta\mathbf{x}|_o = 0 \Rightarrow \mathbf{w} = 0 \Rightarrow G_g = 0$, thus there is no room in this case for the second mechanism (46), (48).) The piece $s(\tau)(\mathbf{uw})\mathbf{u}$ is already included within the gauge transformations, and the last two pieces are just translations, already described too. What seems to be a new piece,

$$\delta\mathbf{Q} = -s(\tau)\mathbf{w}\,,$$

is in fact a combination of a gauge transformation in the $\mathbf{u}$ direction and the transformation obtained in (52), orthogonal to $\mathbf{u}$.

Rigid symmetries with quadratic generators. The original Lorentz transformations with parameters such that $\omega^\mu{}_\nu u^\nu = 0$, $a^\mu + \omega^\mu{}_\nu c^\nu = 0$ yield transformations with quadratic generators for the fluctuations theory. The generator is

$$D^{2h} G_p = P_\mu \omega^\mu{}_\nu Q^\nu$$

with $\omega^\mu{}_\nu u^\nu = 0$, where the operator D^{2h} is defined in phase space as D^{2l} is in tangent space, shown in equation (43). The transformations are

$$\delta Q^\mu = \omega^\mu{}_\nu Q^\nu,$$

again with the rotation parameter ω restricted to the subspace orthogonal to $\mathbf{u}$, i.e. with $\omega^\mu{}_\nu u^\nu = 0$. For example, when $u^\mu = \delta^\mu_0$ is the unit vector along the time direction, the condition $\omega^\mu{}_\nu u^\nu = 0$ becomes $\omega^\mu{}_0 = 0$, giving the transformations (in the index notation $\mu = (0, i)$)

$$\delta Q^0 = 0\,, \qquad\qquad \delta Q^i = \omega^i{}_j\, Q^j,$$

which describe the infinitesimal spatial rotations.

6.2 Yang-Mills theory

In a field theory, the coordinates $q^i(t)$ are exchanged by the fields $\phi^{i,\mathbf{x}}(t) := \phi^i(t,\mathbf{x})$, where the spatial point $\mathbf{x}$ plays the role of a continual index. In consequence, summations $\sum_i$ become integrals $\sum_i \int d\mathbf{x}$, derivatives $\frac{\partial}{\partial q^i(t)}$ become functional variations $\int d\mathbf{x}\, \frac{\delta}{\delta\phi^{\mathbf{i}}(\mathbf{t},\mathbf{x})}$, while all other variables, for example momenta $p_i(t)$, become densities $\pi_i(t,\mathbf{x})$. The Lagrangian density $\mathcal{L}(\phi, \partial\phi)$ depends on the fields ϕ, velocities $\dot\phi$ and spatial gradients $\frac{\partial\phi}{\partial\mathbf{x}}$. The Lagrangian is then $L(\phi,\dot\phi) = \int d\mathbf{x}\, \mathcal{L}(\phi, \partial\phi)$. All boundary terms are neglected (the fields vanish at the boundary fast enough)

and therefore the Lagrangian density $\mathcal{L}$ is determined up to a total divergence (and similarly for a Hamiltonian density $\mathcal{H}$). The basic Poisson bracket is $\{\phi^i(x), \pi_j(x')\}_{t=t'} = \delta^i_j\, \delta(\mathbf{x}-\mathbf{x}')$, but writing the arguments x, x' and δ-function will be omitted for the sake of simplicity.

Consider the Yang-Mills (YM) field theory described by the Lagrangian (density)

$$\mathcal{L}(A, \partial A) = -\frac{k}{4}\, F^a_{\mu\nu} F^{\mu\nu}_a\,. \tag{53}$$

The gauge field $A^a_\mu(t, \mathbf{x}) =: A^a_\mu(x)$ and the associated field strength $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^a_{bc}\, A^b_\mu A^c_\nu$ depend on the coordinates $x^\mu = (x^0, x^i) = (t, \mathbf{x})$ of Minkowski spacetime. The constant k (usually denoted by $1/g^2_{YM}$) is dimensionless and positive. The indices $a, b, \dots$ label the Lie generators of a non-Abelian (semi-simple) Lie group with the structure constants f_{abc} and the Cartan metric g_{ab}.[11] The YM Lagrangian is invariant under the gauge transformations $\delta A^a_\mu(x) = D_\mu \alpha^a(x) \equiv \partial_\mu \alpha^a + f^a_{bc}\, A^b_\mu \alpha^c$. The Euler-Lagrange e.o.m.

$$[\mathcal{L}]^\mu_a := \frac{\partial \mathcal{L}}{\partial A^a_\mu} - \partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu A^a_\mu)} = k\, D_\nu F^{\nu\mu}_a \tag{54}$$

have singular Hessian $W^{\mu\nu}_{ab} = k\, g_{ab}\, (\eta^{\mu\nu} + \eta^{\mu 0}\eta^{\nu 0})$, with zero modes $(\gamma_b)^a_\mu = \delta^0_\mu \delta^a_b$, leading to the primary Lagrangian constraints $[\mathcal{L}]^0_a = k\, D_i F^{i0}_b \simeq 0$.

Defining the canonical momenta

$$\hat{\pi}^\mu_a(A, \partial A) = \frac{\partial \mathcal{L}}{\partial \dot{A}^a_\mu} = -k\, F^{0\mu}_a\,, \tag{55}$$

we pass to Hamiltonian formalism and find primary (π^0_a) and secondary (θ_a) constraints

$$\pi^0_a \simeq 0\,, \qquad\qquad \theta_a \equiv \dot{\pi}^0_a = D_i \pi^i_a \simeq 0\,, \tag{56}$$

with the Dirac Hamiltonian density

$$\mathcal{H}_D = \frac{1}{2k}\, \pi^i_a \pi^a_i + \frac{k}{4}\, F^a_{ij}\, F^{ij}_a - A^a_0\, \theta_a + \lambda^a \pi^0_a\,. \tag{57}$$

Lagrange multipliers $\lambda^a(x)$ can be determined in tangent space (see Appendix B) from Hamilton-Dirac's equations, $\lambda^a(A, \dot{A}) = \dot{A}^a$. The components A^a_0 play the role of Lagrange multipliers for secondary constraints θ_a. These constraints do not evolve in time since $\dot{\theta}_a = -f^c_{ab}\, A^b_0\, \theta_c \simeq 0$, and thus there are no new constraints. (In calculation, the identity $[D_i, D_j]v^a = f^a_{bc}\, F^b_{ij}\, v^c$ was used.) All constraints are First Class and the only nontrivial brackets close non-Abelian algebra

$$\{\theta_a, \theta_b\} = f^c_{ab}\, \theta_c\,. \tag{58}$$

The canonical generator

$$G[\alpha] = \int d\mathbf{x}\, \Big(D_0 \alpha^a \pi^0_a - \alpha^a \theta_a \Big) \tag{59}$$

induces gauge transformations

$$\delta A^a_\mu = \{A^a_\mu, G[\alpha]\} = D_\mu \alpha^a\,, \qquad\qquad \delta \pi^\mu_a = \{\pi^\mu_a, G[\alpha]\} = -f^c_{ab}\, \alpha^b \pi^\mu_c\,. \tag{60}$$

The transformation law for π^a_μ is consistent with $\delta\hat{\pi}^a_\mu$ obtained from (55).

[11] Minkowski metric $\eta_{\mu\nu}$ raises and lowers spacetime indices, and the Cartan metric g_{ab} raises and lowers group indices.

6.2.1 Fluctuations theory

Consider now the small fluctuations Q expanded as $A = \bar{A} + \epsilon Q$ around a solution $\bar{A}$ of the e.o.m. (54). Then the corresponding field strength expands as

$$F^a_{\mu\nu} = \bar{F}^a_{\mu\nu} + \epsilon \left(\bar{D}_\mu Q^a_\nu - \bar{D}_\nu Q^a_\mu \right) + \epsilon^2 f^a_{bc}\, Q^b_\mu Q^c_\nu \,,$$

where all hatted operators denote these operators evaluated at $\bar{A}$. Therefore, we find the Lagrangian for fluctuations theory

$$\tilde{\mathcal{L}}(Q, \partial Q) = -\frac{k}{2} \left[\bar{D}^\mu Q^\nu_a \left(\bar{D}_\mu Q^a_\nu - \bar{D}_\nu Q^a_\mu \right) + f^a_{bc}\, \bar{F}^{\mu\nu}_a Q^b_\mu Q^c_\nu \right] . \tag{61}$$

The canonical formulation of (61) can be obtained directly from the original Hamiltonian analysis, by expanding it as $A = \bar{A} + \epsilon Q$ and $\pi = \bar{\pi} + \epsilon P$ around a solution $\bar{A}$, $\bar{\pi}$ of the canonical e.o.m. Here the basic bracket is $\{Q, P\}^{\tilde{}} = 1$. Linear terms of the original constraints (56) give the primary (P^0_a) and secondary ($\tilde{\theta}_a$) constraints in the fluctuations theory

$$P^0_a \simeq 0 \,, \qquad\qquad \tilde{\theta}_a = \bar{D}_i P^i_a + \, f^c_{ab}\, Q^b_i \,\bar{\pi}^i_c \simeq 0 \,.$$

The Dirac Hamiltonian (57) with the multipliers $\dot{A}^a_0 = \frac{d\bar{A}^a_\mu}{dt} + \epsilon \dot{Q}^a_0 \equiv \bar{\lambda} + \epsilon\tilde{\lambda}$ expands as

$$\mathcal{H}_D\,(A, \pi, \lambda) = \mathcal{H}(\bar{A}^a_\mu, \bar{\pi}^\mu_a) + \epsilon\,(\frac{d\bar{A}^a_\mu}{dt}\, P^\mu_a - \frac{d\bar{\pi}^\mu_a}{dt})\, Q^a_\mu + \epsilon^2 \tilde{\mathcal{H}}_D(Q, P, \tilde{\lambda}) + \mathcal{O}(\epsilon^3) \,,$$

where $\tilde{\mathcal{H}}_D = \tilde{\mathcal{H}} + \tilde{\lambda}^a P^0_a$. The canonical Hamiltonians of the fluctuations theory is

$$\tilde{\mathcal{H}} = \frac{1}{2k}\, P^i_a P^a_i + \frac{k}{2}\, \bar{D}^i Q^j_a \left(\bar{D}_i Q^a_j - \bar{D}_j Q^a_i \right) + \frac{k}{2}\, f^a_{bc}\, \bar{F}^{ij}_a Q^b_i Q^c_j + f^a_{bc}\, \bar{A}_{0a} P^j_b Q^c_j - Q^a_0\, \tilde{\theta}_a \,.$$

This result can also be obtained directly from $\tilde{\mathcal{H}} = \hat{P}Q - \mathcal{L}$. We find that the constraint algebra is Abelianized,

$$\{\tilde{\theta}_a, \tilde{\theta}_b\}^{\tilde{}} = f^c_{ab}\, \bar{\theta}_c \equiv 0 \,,$$

as expected from (27).

Consider now symmetries of the fluctuations theory. The gauge generator (59) expands as $G[\alpha] = \epsilon \tilde{G}[\alpha] + \epsilon\, \tilde{\tilde{G}}[\alpha]$, where the linear ($\tilde{G}$) and quadratic ($\tilde{\tilde{G}}$) generators are

$$\tilde{G}[\alpha] \;=\; \int d\mathbf{x}\, (\bar{D}_0 \alpha^a P^0_a - \alpha^a \tilde{\theta}_a) \,, \tag{62}$$

$$\tilde{\tilde{G}}[\alpha] \;=\; -\int d\mathbf{x}\, f^c_{ab}\, \alpha^a Q^b_\mu P^\mu_c . \tag{63}$$

Linear generator induces Abelianized gauge symmetries $\delta Q^a_\mu = \bar{D}_\mu \alpha^a$, in agreement with (47). $\tilde{\tilde{G}}[\alpha]$ is a generator of rigid symmetries in fluctuations theory only if there are such parameters α^a and vacuums $\bar{A}^a$, for which $\delta \bar{A}^a_\mu = \bar{D}_\mu \alpha^a = 0$. Then the transformations

$$\tilde{\delta} Q^a_\mu = f^a_{bc}\, Q^b_\mu \alpha^c \,, \qquad\qquad \tilde{\delta} P^\mu_a = -f^c_{ab}\, P^\mu_c \alpha^b \,,$$

leave $\tilde{\mathcal{H}}$ invariant. Similarly, $\tilde{\delta}Q$ leaves $\tilde{\mathcal{L}}$ invariant. Indeed, under the transformations $\tilde{\delta}Q^a_\mu = f^a_{bc}Q^b_\mu\alpha^c$, the Lagrangian (61) changes as

$$\tilde{\delta}\tilde{\mathcal{L}} = -k\, f_{abc}\left[\left(\bar{D}_\mu Q^a_\nu - \bar{D}_\nu Q^a_\mu\right) Q^{\nu b} + Q^a_\mu\, Q^b_\nu\, \bar{D}^\nu\right] \bar{D}^\mu \alpha^c\,,$$

where the identity $f_{abc}\bar{D}_{[\mu}Q^a_{\nu]}\bar{D}^{[\mu}Q^{\nu]b} \equiv 0$, the Jacobi identity $f^c_{a[b}\, f_{de]c} \equiv 0$ and $[\bar{D}_\mu, \bar{D}_\nu]\alpha^c = f^c_{ae}\bar{F}^a_{\mu\nu}\alpha^e$ have been used.[12] In that way, for the backgrounds for which $\bar{D}\alpha^a = 0$, the fluctuations Lagrangian becomes invariant, $\tilde{\delta}\tilde{\mathcal{L}} = 0$. The condition $\bar{D}\alpha^a = 0$ on the gauge parameter α^a is not trivial. The existence of a solution depends on the topology of a manifold where the YM theory is defined and on the boundary conditions for α^a, as well as on the properties of the background $\bar{A}$ (such as its possible winding numbers, etc.). Additionally, α^a has to be globally defined.

7 Conclusions

This work is dedicated to the study of the dynamics of small fluctuations oscillating around a classical solution of a gauge theory. This system, where the fluctuation is a fundamental field, is described by the explicitly time-dependent quadratic Lagrangian or Hamiltonian. We show that in First Order systems, that is, ordinary tangent space or phase space formulations, it is permitted to choose freely between the Lagrangian and Hamiltonian formalism, with the certainty that the same result will be reached at the end, assuming that the conditions (i)-(iii) at the begining of Section 3, have been fulfilled. We show that in such a case, the Legendre transformation commutes with the transformation which maps the original Lagrangian to the quadratic one, at first order in the fluctuations expansion. In fact, the mismatch in Lagrangian and Hamiltonian descriptions occurs in higher orders in the expansion parameter ϵ, but it does not affect the consistency of neither of these two descriptions.

Other results of our analysis are the following. While the fluctuations Lagrangian is defined as the quadratic term of the original Lagrangian, the fluctuations canonical Hamiltonian contains, apart from the quadratic term coming from the original canonical Hamiltonian, also the contribution of the quadratic part of primary constraints and the corresponding Lagrange multipliers computed for the classical solution under consideration.

Furthermore, we prove that, under the assumptions made in Section 3, this mapping, or "linearization" of the theory (since the equations of motion become linearized by it), entirely keeps the structure of the original theory in the fluctuations theory as well. For example, the class of constraints (First or Second) does not change after the linearization, and the structure of the constraint algorithm remains the same. Since the linearization does not change the number of First and Second class constraints, it follows that the number of physical degrees of freedom in both theories is the same.

[12] $[\cdots]$ acts on the indices inside the bracket as the antisymmetrization.

We address in particular the issue of Noether symmetries. As regards the gauge ones, we find that the First Class constraints in the fluctuations theory always correspond to Abelian gauge symmetries, since the non-Abelian symmetry cannot be described by quadratic Lagrangians. Another interesting outcome is that the choice of the background may influence the expression of the rigid Noether symmetries in the fluctuations theory. One part of these symmetries (the one which mimics those of the original theory) is generated by the linear terms of the fluctuations expansion of the original generators. If, however, it happens that some background is preserved by a subset of the original rigid symmetries, then the fluctuations theory exhibits rigid symmetries coming from the quadratic powers in the fluctuations of the original generator. In supersymmetric theories this is the way for instance in which the symmetries preserved by a BPS state are realized in the fluctuations theory.

As mentioned above, our results are reliable for the systems with constant Hessian and around non-degenerate solutions. For the systems with degenerate solutions, the linear approximation is not applicable any longer. For example, a degenerate solution q^o can lead to the ineffective constraints (of the type $(q-q^o)^2 \simeq 0$). After the linearization, these constraints vanish, what effectively leads to the increase in the number of degrees of freedom. One example of a such degenerate background in Chern-Simons supergravity is presented in ref. [1] and they are treated in Hamiltonian formalism in ref. [12]. In Lagrangian formalism, they have not been studied yet.

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A Identities with the bracket expansions

Here we give some results used in the main text. For generic expansions of functions $f(q,p;t)$, $g(q,p;t)$,

$$\begin{aligned} f(q,p;t) &= f_o(t)+\epsilon f_1(Q,P;t)+\epsilon^2 f_2(Q,P;t)+\mathcal{O}(\epsilon^3) \\ g(q,p;t) &= g_o(t)+\epsilon g_1(Q,P;t)+\epsilon^2 g_2(Q,P;t)+\mathcal{O}(\epsilon^3)\,, \end{aligned} \tag{64}$$

(where $f_o(t)=f(q^o(t),p^o(t);t)$, $g_o(t)=g(q^o(t),p^o(t);t)$ and $f_1=D^hf$, $g_1=D^hg$) the following relations hold

$$\{f_1,\, g_1\}^{\tilde{}} = \{f,\, g\}|_o\,, \tag{65}$$

$$D^h\{f,\, g\} = \{f_2,\, g_1\}^{\tilde{}} + \{f_1,\, g_2\}^{\tilde{}}\,. \tag{66}$$

The proofs are immediate, using (64) and the corresponding expansion for the bracket $\{f,\, g\}$.

Explicit time dependence has a different meaning in the original theory and in the fluctuations theory because the change of variables $q \to q^o + \epsilon Q$ is time dependent —q^o is in fact the trajectory $q^o(t)$. In particular we have, in the canonical formalism (similar results hold in the tangent space formulation),

$$\begin{aligned}
\frac{\partial^f}{\partial Q} &= \epsilon \frac{\partial}{\partial q} \\
\frac{\partial^f}{\partial P} &= \epsilon \frac{\partial}{\partial p} \\
\frac{\partial^f}{\partial t} &= \frac{\partial}{\partial t} + \dot{q}^o \frac{\partial}{\partial q} + \dot{p}^o \frac{\partial}{\partial p}\,,
\end{aligned} \tag{67}$$

where the superscript ^{f} stands for the fluctuation variables and is conventionally omitted in the text. Note that

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \dot{q}\frac{\partial}{\partial q} + \dot{p}\frac{\partial}{\partial p} + ... = \frac{\partial^f}{\partial t} + \dot{Q}\frac{\partial^f}{\partial Q} + \dot{P}\frac{\partial^f}{\partial P} + ...\,,$$

(where the dots indicate higher-order tangent structures like $\ddot{q}\frac{\partial}{\partial \dot{q}} + \ddot{p}\frac{\partial}{\partial \dot{p}}$ etc.) as it must be.

One can also derive the following result used in the text,

$$[D^h,\, \frac{\partial}{\partial t}]f = \{P\dot{q}^o - Q\dot{p}^o,\, f_2\}^{\tilde{}}\,. \tag{68}$$

Obviously the l.h.s. must be understood as $D^h \circ \frac{\partial}{\partial t} - \frac{\partial^f}{\partial t} \circ D^h$. The proof of (68) is

$$\begin{aligned}
[D^h,\, \frac{\partial}{\partial t}]f &= Q\frac{\partial f}{\partial q \partial t}|_o + P\frac{\partial f}{\partial p \partial t}|_o - \frac{\partial}{\partial t}(Q\frac{\partial f}{\partial q}|_o + P\frac{\partial f}{\partial p}|_o) \\
&= -\Big(Q\frac{\partial f}{\partial q \partial q}|_o \dot{q}^o + Q\frac{\partial f}{\partial q \partial p}|_o \dot{p}^o + P\frac{\partial f}{\partial p \partial q}|_o \dot{q}^o + P\frac{\partial f}{\partial p \partial p}|_o \dot{p}^o\Big) \\
&= -\dot{q}^o\frac{\partial f_2}{\partial Q} - \dot{p}^o\frac{\partial f_2}{\partial P} = \{P\dot{q}^o - Q\dot{p}^o,\, f_2\}^{\tilde{}}\,.
\end{aligned} \tag{69}$$

B Multipliers in Hamilton-Dirac equations of motion

The Hamilton-Dirac e.o.m. are

$$\begin{aligned}
\dot{q} &= \frac{\partial H}{\partial p} + \lambda^\mu \frac{\partial \phi_\mu}{\partial p}\,, \\
\dot{p} &= -\frac{\partial H}{\partial q} - \lambda^\mu \frac{\partial \phi_\mu}{\partial q}\,, \\
0 &= \phi_\mu^{(0)}(q,p)\,,
\end{aligned} \tag{70}$$

where the λ^μ are in principle arbitrary functions of time (in fact they may also be arbitrary functions of the q and p variables in the Hamiltonian approach, but this is not relevant for our discussion). Not any choice for these functions is allowed in the formalism, as it is seen when one performs the Dirac-Bergman constraint algorithm, where eventually some of these functions become determined in phase space whereas some others stay completely arbitrary and in fact describe the gauge freedom (local symmetries) contained in the dynamics.

In spite of their initial arbitrariness in phase space, it is interesting to notice that one can always determine these arbitrary functions λ^μ as definite functions in tangent space. To this end one must first use the first equation in (70) and apply the pullback $p \to \hat{p}$; then, since the matrix $\frac{\partial \phi_\mu}{\partial p}$ has maximum rank, the algebraic equation for the λ^μ's,

$$\dot{q} = \mathcal{F}L^* \frac{\partial H}{\partial p} + \lambda^\mu \mathcal{F}L^* \frac{\partial \phi_\mu}{\partial p}\,, \tag{71}$$

can be solved for all λ^μ as functions $\lambda^\mu_{def}(q,\dot{q})$ defined in tangent space. The rationale for this construction is as follows: if for some given set of arbitrary functions $\lambda^\mu(t)$ we obtain a solution $q(t), p(t)$ of (70), then $\lambda^\mu_{def}(q(t),\dot{q}(t)) = \lambda^\mu(t)$. In fact the e.o.m. (70) are completely equivalent to the following e.o.m.

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p} + \lambda^\mu_{def}(q,\dot{q}) \frac{\partial \phi_\mu}{\partial p}\,, \\ \dot{p} &= -\frac{\partial H}{\partial q} - \lambda^\mu_{def}(q,\dot{q}) \frac{\partial \phi_\mu}{\partial q}\,, \\ 0 &= \phi^{(0)}_\mu(q,p)\,. \end{aligned} \tag{72}$$

Thus one can either work with arbitrary functions $\lambda^\mu(t)$, as in (70), or just consider that the e.o.m. are (72). In (72) we see that the unknown $\dot{q}$ appears not only in the l.h.s., but also in the r.h.s. The impossibility to write (72) in normal form[13] is signaling the possible presence of gauge freedom.

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